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# SIMPLE PROOF OF THE EXISTENCE OF RESTRICTED RAMSEY GRAPHS BY MEANS OF A PARTITE CONSTRUCTION

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By means of a partite construction we present a short proof of the Galvin Ramsey property of the class of all finite graphs and of its strengthening proved in [5]. We also establish a generalization of those results. Further we show that for every positive integer $m$ there exists a graph $H$ which is Ramsey for $K_m$ and does not contain two copies of $K_m$ with more than two vertices in common.

## 1. Introduction

The following generalization of the well-known theorem of Ramsey [10] was proved independently by Erdős, Hajnal and Pósa [2], Deuber [1] and Rödl [11].

**Theorem A.** *Given a finite graph* $G=(V, E)$ *there exists a graph* $R(G)=(W, F)$ *such that for every partition* $F=F_1\cup F_2$ *there are* $V'\subset W$ *and* $i\in\{1, 2\}$ *such that if we denote* $E'=\{e\in F\colon e\subset V'\}$ *then* $(V', E')$ *is isomorphic to* $(V, E)$ *and* $E'\subset F_i$.

In the following we shall call such a graph $R(G)$ a *Ramsey graph for G*. Theorem A was strengthened in [5], extending an earlier result of J. Folkman [4] as follows:

**Theorem B.** *For every graph G there exists a Ramsey graph such that* $\mathrm{cl}\,(G)=\mathrm{cl}\,(H)$.

(Here $\mathrm{cl}\,(G)$ and $\mathrm{cl}\,(H)$ denotes the size of a *maximal clique in* $G$ and $H$ respectively.)

The original proofs of Theorems A and B were not easy. A relatively simple proof of the Theorem A was given in [7]. The purpose of this paper is to introduce a method by means of which both above theorems can be proved quite easily. We also give a solution to a question of P. Erdős [3] (see Theorem C below) and we prove Theorem D which extends both Theorems B and C.

**Theorem C.** *Let* $m\geqq 3$ *be a fixed positive integer. Then there exists a Ramsey graph H for* $K_m$ *such that any two subgraphs* $K, K'$ *of H isomorphic to* $K_m$ *intersect in at most two points.*

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Let $\mathcal{F}$ be a set of graphs. Put Forb $(\mathcal{F})=\{H$: $F$ fails to be an induced subgraph of $H$ for any $F\in\mathcal{F}\}$.

**Theorem D.** *Let $\mathcal{F}$ be a set of 3-chromatically connected graphs, (i.e. the subgraph of $F$ induced on any vertex cut set has chromatic number $\geqq 3$ for any $F\in\mathcal{F}$). Then for every $G\in$ Forb $(\mathcal{F})$ there exists a Ramsey graph $H\in$ Forb $(\mathcal{F})$.*

Our method uses only amalgamation of graphs and is a refinement of [6]. It is presented here in its simplest form. Further applications of this method may be found in the survey papers [8] and [9].

## 2. Basic notions and notation

By a graph we shall understand a couple $G=(V, E)$ where $V=V(G)$ is a set of vertices and $E=E(G)\subset[V]^2$ is a set of edges. (The symbol $[V]^k$ denotes the set of all $k$-element subsets of $V$).

In our construction the following notions are important. Let $(V_i)_{i=1}^r$ be a system of pairwise disjoint sets and let $E\subset\left[\bigcup_{i=1}^r V_i\right]^2$ such that $E\cap[V_i]^2=\emptyset$ for all $i=1, 2, \ldots, r$. Then the couple $G=\left((V_i)_{i=1}^r, E\right)$ is called an *r-partite graph.* It will be convenient to write $V_i=V_i(G)$. Let $G=\left((V_i)_{i=1}^r, E\right)$, $H=\left((W_i)_{i=1}^r, F\right)$ be two $r$-partite graphs. We say that $G$ is an *induced subgraph* of $H$ if $V_i\subset W_i$ for every $i=1, 2, \ldots, r$ and the graph $\left(\bigcup_{i=1}^r V_i, E\right)$ is an induced subgraph of $\left(\bigcup_{i=1}^r W_i, F\right)$. We shall denote this relation by $G\leqq H$.

## 3. The Bipartite Lemma

In the proof we shall use the following:

**Lemma.** *For every bipartite graph $B=(V_1, V_2, E)$ there exists a Ramsey bipartite graph $R(B)=(W_1, W_2, F)$, i.e. the following holds: For every partition $F=F_1\cup F_2$ there exist $V_1'\subset W_1$, $V_2'\subset W_2$ and $i\in\{1, 2\}$ such that if we denote $E'=\{e\in F$: $e\subseteqq V_1'\cup V_2'\}$ then $E'\subseteqq F_i$ and $(V_1', V_2', E')$ is isomorphic to $(V_1, V_2, E)$.*

This Lemma is folklore. For the sake of completeness we include a simple proof Let $|V_1|=k_1$, $|V_2|=k_2$. It is easy to see that $(V_1, V_2, E)$ is an induced subgraph of $(X, [X]^{k_1}, \bar{E})$, $|X|\geqq k_2+2k_1$, where

$$\{x, A\}\in\bar{E} \quad \text{iff} \quad x\in A.$$

Consequently we may assume $(V_1, V_2, E)=(X, [X]^{k_1}, \bar{E})$ without loss of generality. Put $k=2(k_1-1)+1$ and let $Y$ be a set with cardinality at least $r\left(k, 2\binom{k}{k_1}, k|X|\right)$ where $r(a, b, c)$ is the Ramsey number for the partition of $a$-tuples into $b$ parts with homogenous $c$-set. Let $W_1=Y$, $W_2=[Y]^k$ and $F=\{\{y, K\}, y\in K\}$ where $y\in Y$ and $K\in[Y]^k$. Using Ramsey's theorem it can be shown easily that the bipartite graph $(W_1, W_2, F)$ has the desired properties. ∎

## 4. Proof of Theorem A — a partite construction

Let $G$ be a graph with $m$ vertices and let $H=K_r$ (the complete graph with $r$ vertices), where $r=r(2, 2, m)$.

For each $A\in[V(H)]^m$ fix one graph with vertex set $A$, isomorphic to $G$. Let $\{G_1, G_2, \ldots, G_{\binom{r}{m}}\}$ be a system of all such graphs. Put

$$V(H) = \{v_1, v_2, \ldots, v_r\}$$
$$E(H) = \{e_1, e_2, \ldots, e_R\}, \text{ where } R = \binom{r}{2}.$$

Define inductively an $r$-partite graph $P^n$ for all $n\leqq R$, as follows:

$$P^0 = ((V_i^0)_{i=1}^r, E^0)$$

where

$$V_i^0 = \{(v_i, j),\ j \leqq R\}$$

and $\{(v_i, j), (v_{i'}, j')\}\in E^0$ if and only if $j=j'$ and $\{v_i, v_{i'}\}\in E(G_j)$. Suppose we have defined the $r$-partite graph $P^n=((V_i^n)_{i=1}^r, E^n)$. Put $e_{n+1}=\{v_{x_1}, v_{x_2}\}$ and let $B$ be a bipartite subgraph of $P^n$ induced on a set $V_{x_1}^n\cup V_{x_2}^n$. Let $R(B)$ be a bipartite graph which is Ramsey for $B$ (the existence of which is ensured by the Lemma). Denote by $q$ the number of induced subgraphs of $R(B)$ which are isomorphic to $B$. Explicitly, let $B_1, B_2, \ldots, B_q$ be all the induced subgraphs of $R(B)$ which are isomorphic to $B$. For each $i\leqq q$ let $\varphi_i$: $B_i\to B$ be the natural inclusion. Put

$$V(P^{n+1}) = \bigcup_{i=1}^{r} V_i^{n+1}$$

where $V_i^{n+1}=\bigcup_{j\leqq q}(V_i^n\times\{j\})$ for $i\neq x_1, x_2$; $V_{x_i}^{n+1}=V_i(R(B))$ for $i=1, 2$. Denote by $\psi_j$: $V(P^n)\to V(P^{n+1})$ the 1—1 mapping defined by

$$\psi_j(v) = \varphi_j(v) \quad \text{for} \quad v\in V(B),$$
$$\psi_j(v) = (v, j) \quad \text{for} \quad v\notin V(B).$$

We say $\{v_1, v_2\}\in E(P^{n+1})$ if and only if there exist $j\leqq q$ and $\{u_1, u_2\}\in E(P^n)$ such that $v_1=\psi_1(u_1)$ and $v_2=\psi_2(u_2)$. We introduce the following notation. Let $e\in E(P^R)$. We say that the edge $e'=\{i, j\}$ is a projection of the edge $e$ if $|e\cap V_i|=1$ and $|e\cap V_j|=1$.

**Fact.** *$P^R$ is Ramsey for $G$.*

**Proof.** The following holds:

$$(*) \qquad P^0 \xrightarrow{e_1} P^1 \xrightarrow{e_2} \ldots \xrightarrow{e_s} P^s,$$

where the symbol $P^n \xrightarrow{e_{n+1}} P^{n+1}$ denotes the following: if we split the edges of $P^{n+1}$ with projection $e^{n+1}$ into two parts then there exists an $r$-partite graph $\bar{P}^n$, $\bar{P}^n\leqq P^{n+1}$ isomorphic to $P^n$ such that all edges of $\bar{P}^n$ with projection $e_{n+1}$ are in one of the classes of the partition. From $(*)$ (using backward induction from $R$ to 1) it follows that the graph $P^R=((V_i^R)_{i=1}^r, E)$ has the following property: For every partition

$E^R = E_1 \cup E_2$ there exists $\bar{P}^0 \simeq P^0$, $\bar{P}^0 \leqq P^R$ such that any two edges of $P^0$ with the same projection are in the same partition class. Using the construction of $P^0$ it follows that there exists an induced subgrapx of $P^R$ isomorphic to $G$ with all edges belonging to one of the classes of the partition. ▮

## 5. Corollaries

The following are easy corollaries of the above construction.

**Proof of Theorem B.** Clearly $\mathrm{cl}\,(G) = \mathrm{cl}\,(P^0)$ and $\mathrm{cl}\,(P^i) = \mathrm{cl}\,(P^{i+1})$ for every $i < \binom{r}{2}$ and hence $\mathrm{cl}\,(G) = \mathrm{cl}\,(P^R)$. ▮

**Proof of Theorem C.** Apply the above partite construction for $G = K_m$. Clearly $P^0$ does not contain two graphs $K, K'$ isomorphic to $K_m$ intersecting in more than two vertices. Moreover if $P^i$ does not contain two graphs isomorphic to $K_m$ intersecting in more than two vertices then $P^{i+1}$ also has this property, as every triangle (i.e. $K_3$) in $P^{i+1}$ belongs to exactly one copy of $P^i$. ▮

**Proof of Theorem D.** Apply the above construction. Observe that $P^i \in \mathrm{Forb}\,(\mathscr{F})$ implies $P^{i+1} \in \mathrm{Forb}\,(\mathscr{F})$. ▮

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